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Partially directed site percolation on the square and triangular lattices

H O Mártin† and J Vannimenus

Groupe de Physique des solides de l'Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cedex 05, France

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Abstract. Using the phenomenological renormalisation (PR) method we obtain the critical probability for partially directed site percolation on the square and triangular lattices ($p_c = 0.6317 \pm 0.006$ and $p_c = 0.5468 \pm 0.005$, respectively). We discuss the corrections to scaling in the calculation of the anisotropy exponent θ ($= \nu_{\parallel}/\nu_{\perp}$) and the critical probability, and we show that it is necessary to work with at least two correction terms. This is in contrast to the usual analysis where only one correction exponent is used in PR calculations.

1. Introduction

It is well known that the introduction of a privileged connectivity direction in the percolation problem leads to a different universality class from that of ordinary percolation (Obukhov 1980, Kinzel 1983), but much more effort has been devoted to the case of fully directed percolation (see figure 1(a)) than to the partially directed one (see figure 1(b)) (Kertész and Vicsek 1980). The first purpose of the present paper is to compute accurately, for the first time to our knowledge, the critical probability for partially directed site percolation on the square and triangular lattices (see figures 1(b) and (c)).

In our calculation we have used the transfer matrix technique and the phenomenological renormalisation (PR) approach introduced by Nightingale (1976, 1982 and references therein) for spin systems and generalised to percolation by Derrida and Vannimenus (1980). The PR is a powerful numerical method and one can frequently obtain more precise results using the PR than with other methods (such as the Monte-Carlo approach, for example). However, one of the difficulties in this method is that one does not in general know the form of the corrections to scaling which enter the extrapolation to the thermodynamic limit, and there is presently some debate about the extrapolation procedures (Privman and Fisher 1983, Privman 1984, Herrmann and Stauffer 1984). The problem of the convergence of the numerical results is more delicate for partially directed problems than for fully directed ones to judge from the work of Nadal *et al* (1982) and Privman and Barma (1984) on directed lattice animals: partially directed percolation turns out to be very instructive in that respect, and it provides a good example to study corrections to scaling.

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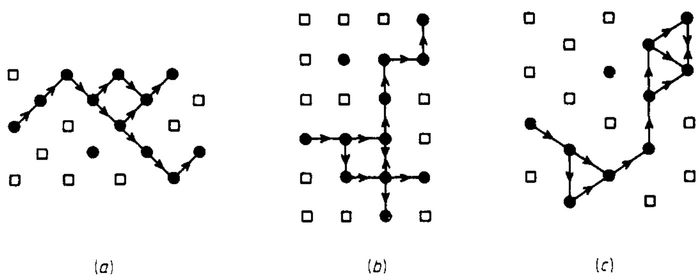

Figure 1. Examples of clusters for (a) fully directed site percolation on the square lattice and (b) and (c) partially directed site percolation on the square and triangular lattices, respectively. The full circles (open squares) denote the sites present (absent). The clusters are the sites present connected by lines.

The discussion of these corrections and of the suitable extrapolation procedure is the second purpose of the present work and is developed in § 3. We find that the corrections-to-scaling exponent is smaller than one (Privman and Fisher 1983, Privman 1984), but that is necessary to keep two correction terms to have a consistent extrapolation for the two lattices studied.

2. Brief review of the method

In the directed percolation problem there are two different correlation lengths, $\xi_\parallel$ and $\xi_\perp$, parallel and perpendicular to the privileged direction, respectively (Kinzel 1983). Close to the critical probability p_c these lengths behave as

$$\xi_\parallel \sim (p_c - p)^{-\nu_\parallel} \qquad \xi_\perp \sim (p_c - p)^{-\nu_\perp} \tag{1}$$

where the lattice sites are present (absent) with probability p $(1-p)$. The calculations follow the methods developed by Derrida and Vannimenus (1980) and Derrida and De Seze (1982): we compute the second largest eigenvalue λ_N of the transfer matrix (the first eigenvalue being identically one) for strips of N rows. The models studied are the square and triangular lattices with periodic boundary conditions which are described in figures 1(*b*) and (*c*) respectively. Taking the privileged direction along the strip, we have (Kinzel and Yeomans 1981)

$$\xi_\parallel^N = -1/\ln(\lambda_N(p)). \tag{2}$$

Now, from PR arguments and working with three strips of width $N-1$, N and $N+1$, we can compute estimates θ_N and p_c^N for the anisotropy exponent $\theta \equiv \nu_\parallel/\nu_\perp$ and for the critical probability by demanding that the following equations hold (for more details see Kinzel and Yeomans 1981):

$$\theta_N = \frac{\ln(\xi_\parallel^N(p_c^N)/\xi_\parallel^{N-1}(p_c^N))}{\ln[N/(N-1)]} = \frac{\ln(\xi_\parallel^{N+1}(p_c^N)/\xi_\parallel^N(p_c^N))}{\ln[(N+1)/N]}. \tag{3}$$

Let us now discuss the problem of the corrections to scaling. As usual (Privman and Fisher 1983), we suppose the scaling form

$$\xi_\parallel^N(p) \simeq N^\theta f(g_p N^{1/\nu_\perp}, g_u N^{-\omega}) \tag{4}$$

for large N where $f(x, y)$ is an analytic function in $x = y = 0$ which takes the corrections due to finite-size effects into account and g_p and g_u are the relevant and irrelevant scaling fields respectively (θ and $\nu_\perp$ are the critical exponents for the infinite system). Aharony and Fisher (1983) (see also Privman and Fisher 1983) have pointed out that these scaling fields are non-linear, and close to the critical point can be expanded as

$$g_p = (p - p_c) + a_p(p - p_c)^2 + \ldots$$
$$g_u = u + b_p(p - p_c) + \ldots \tag{5}$$

where a_p, u and b_p are constants. Let us note that g_p is normalised in such a way as to vanish for $p = p_c$ (Aharony and Fisher 1983, Luck 1984). From equations (3)-(5) and after some algebra we get

$$\theta_N = \theta + N^{-\omega}(A_1 + A_2 N^{-\delta} + \ldots) \tag{6a}$$

$$p_c^N = p_c + N^{-1/\nu_\perp - \omega}(B_1 + B_2 N^{-\delta'} + \ldots) \tag{6b}$$

where A_1, A_2, B_1 and B_2 are constants; for example,

$$A_1 = -(a_1 u/b_0)(\omega\nu_\perp)^2 \qquad B_1 = -(a_1 u/a_0)\omega(1 + \omega\nu_\perp). \tag{7}$$

Here $a_0 = f(0, 0)$, $a_1 = (\partial f(x, y)/\partial y)|_{x=y=0}$ and $b_0 = (\partial f(x, y)/\partial x)|_{x=y=0}$. The exponents δ and δ' are given by

$$\delta = \text{minimum of } \{\omega, 1\} \tag{8a}$$

$$\delta' = \text{minimum of } \{\omega, 2\}. \tag{8b}$$

Let us stress that in equation (4) we have considered (for simplicity) only one irrelevant scaling field, but other irrelevant scaling fields (with exponents ω', ω'', ...; $0 < \omega < \omega' < \omega'' < \ldots$) can appear. In this case equations (6) and (7) hold but

$$\delta = \text{minimum of } \{\omega, 1, (\omega' - \omega)\}$$
$$\delta' = \text{minimum of } \{\omega, 2, (\omega' - \omega)\}. \tag{9}$$

3. Analysis of the results

The results for θ_N and p_c^N obtained on strips of width up to 12 from equations (2) and (3) are shown in tables 1 and 2 for the square and triangular lattices, respectively. We now discuss several procedures to analyse these data and extract an accurate value of p_c.

3.1. Extrapolation with one correction term

We consider first the simplest type of analysis, applied only to the square lattice, and try to fit the values of θ_N and p_c^N using only the dominant correction term in the extrapolation:

$$\theta_N = \theta + AN^{-\omega} \tag{10a}$$

$$p_c^N = p_c + BN^{-\varepsilon}. \tag{10b}$$

From now on, we define the error E of a fit by

$$E = \left(\sum_{N=N_i}^{N_f} (\tilde{x}_N - x_N)^2\right)^{1/2} (N_f - N_i + 1)^{-1} \tag{11}$$

Table 1. Successive approximations θ_N and p_c^N to the anisotropy exponent and the percolation threshold obtained from equations (3) for partially directed site percolation on the square lattice (see figure 1(*b*)) using strips of width $N-1$, N and $N+1$ with periodic boundary conditions. $\tilde{\theta}_N$ and $\tilde{p}_c^N$ are the values obtained using an extrapolation with only one correction term (equations (10)) with $\theta = 1.7052$, $A = -1.5107$, $\omega = 0.53$, $p_c = 0.632\,38$, $B = -0.405\,75$ and $\varepsilon = 1.56$.

N	θ_N	p_c^N	$\tilde{\theta}_N - \theta_N$	$\tilde{p}_c^N - p_c^N$
4	0.9472	0.57909	0.0334	0.00662
5	1.0626	0.59951	−0.0012	−0.00008
6	1.1207	0.60757	$<10^{-4}$	0.00002
7	1.1667	0.61290	−0.0001	−0.00001
8	1.2034	0.61655	$<10^{-4}$	$<10^{-5}$
9	1.2338	0.61920	$<10^{-4}$	0.00001
10	1.2594	0.62120	$<10^{-4}$	$<10^{-5}$
11	1.2813	0.62274	$<10^{-4}$	0.00001

Table 2. Successive approximations θ_N and p_c^N obtained as in table 1 but for the triangular lattice (see figure 1(*c*)).

N	θ_N	p_c^N
4	1.0585	0.50863
5	1.1455	0.52167
6	1.2060	0.52896
7	1.2495	0.53334
8	1.2824	0.53622
9	1.3085	0.53823
10	1.3301	0.53972
11	1.3482	0.54085

where x_N is the true value and $\tilde{x}_N$ is the approximate one. To obtain the constants (such as θ and A in equation (10*a*)) in the extrapolation formulae, we demand that E be a minimum for fixed values of the correction exponents (such as ω in equation (10*a*)). As is seen in table 1, a very good fit can be obtained with $\theta = 1.7052$, $\omega = 0.53$, $p_c = 0.632\,38$ and $\varepsilon = 1.56$. Note that the values of ω and ε agree rather well with the relation $\varepsilon = \omega + 1/\nu_\perp$ given by equation (6) (using $\nu_\perp = 1.1$ (see Kinzel 1983) and $\omega = 0.53$ one obtains $\varepsilon = 1.44$). But this agreement is misleading because the value of the anisotropy exponent is too large: on general field-theoretic grounds, one expects θ to have the same value, $\theta \simeq 1.576$ (Kinzel 1983), as in fully directed percolation. Even if the square lattice were for some reason a special case, one would expect anisotropy effects to be weaker and θ to be *smaller* for partially than for fully directed percolation. The value 1.7052 is well out of the error bars quoted in previous works (for the fully directed case): $1.543 \leq \theta \leq 1.590$ from series expansions (De'Bell and Essam 1983), $1.564 \leq \theta \leq 1.581$ from the quantum reggeon spin model (Brower *et al* 1978, Cardy and Sugar 1980) and $1.567 \leq \theta \leq 1.585$ using the best values presently available for the exponents $\nu_\parallel$ and $\nu_\perp$ (Kinzel 1983). This shows that it can be dangerous to make the extrapolation of equation (10*a*) without any knowledge about the corrections to scaling.

3.2. Confrontation of two lattices

If one still assumes that nothing is known about the value of the anisotropy exponent, another lattice can be used in order to check the result obtained. For the triangular lattice we find a good fit to equations (10) with $\theta = 1.5651$ and $p_c^N = 0.546\,92$, using $\omega = 0.83$ and $\varepsilon = 1.79$. We note that the results contradict the previous ones on the square lattice because the values of θ, ω and ε are quite different, whereas one expects the correction exponents ω and ε to be the same for all partially directed lattices. In figure 2 we plot the error E (defined by equation (11)) in the anisotropy exponent obtained using equation (10a) as a function of ω. It is not possible to find a value of ω for which good fits to equation (10a) are obtained for the square and triangular lattices at the same time.

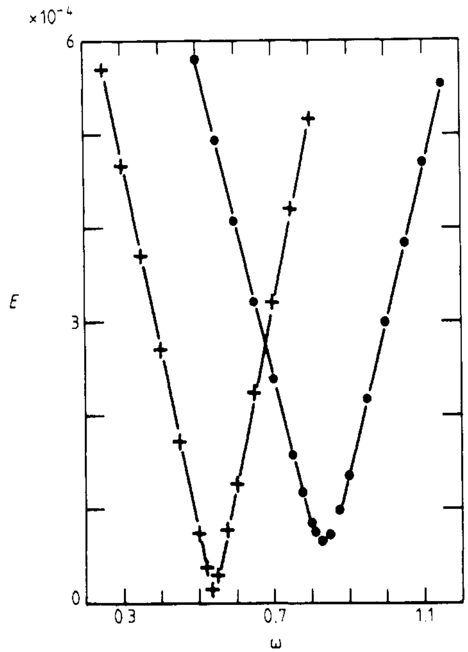

Figure 2. The error E (defined by equation (11)) in the extrapolation using one correction term for the anisotropy exponent (equation (10a)) plotted as a function of the leading correction exponent ω. The crosses correspond to the square lattice and the circles to the triangular one. The fits are performed with $N_i = 6$ and $N_f = 11$.

3.3. Extrapolation with two correction terms

At this stage there are two possibilities in order to make compatible extrapolations: either to work with much larger values of N (which is usually impractical), or to take higher-order terms in the extrapolation into account. We have used this second method, keeping two terms in equations (6). If one assumes no knowledge of the anisotropy exponent, one can ask for the value of ω (δ and δ' are given by equations (8)) for which one obtains the same value θ for the square and triangular lattices. Using $\nu_\perp = 1.1$ (Kinzel 1983), this gives $\omega \simeq 0.96$, and for this value of ω we obtain θ(square) = 1.5428, θ(triangular) = 1.5432, p_c(square) = 0.631 15 and p_c(triangular) = 0.546 59. The value of θ now obtained is much better than the first estimation for the square lattice.

In order to obtain a better estimation of the critical probability, we now fix θ in equation (6a) to the accepted value ($\theta = 1.576$ using the best available values of $\nu_\parallel$ and $\nu_\perp$ (Kinzel 1983)). Following the general method described above, we minimise the error E in the fit of the anisotropy exponent for a given value of the correction exponent ω using equations (6a) and (8a). This error is plotted in figure 3 as a function of ω for the square and triangular lattices. Considering both lattices at the same time, we find that $\omega \simeq 0.8$ is a reasonable value. In figure 4 we show the extrapolated value of p_c as a function of ω for both lattices. Assuming $0.65 \leqslant \omega \leqslant 0.95$, we obtain as our final results

$$p_c(\text{square}) = 0.6317 \pm 0.006$$
$$p_c(\text{triangular}) = 0.5468 \pm 0.005. \tag{12}$$

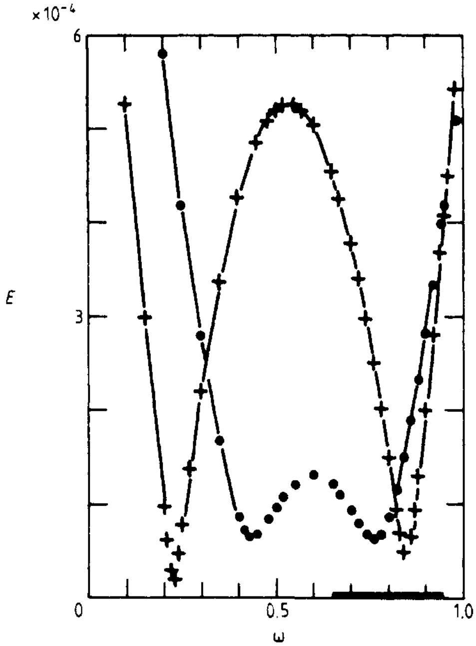

Figure 3. As figure 2 but using two correction terms with the value of θ fixed to $\theta = 1.576$ in equation (6a) and δ given by equation (8a).

The last error bar is greater than the one obtained from the plot of figure 4 because it takes other analyses of equation (11) with different values of N_i and N_f into account.

These results were obtained using equations (8), that is assuming that there is only one irrelevant scaling field. Using the more general equations (9), together with the criterion of minimum error in the anisotropy exponent, we have checked that the introduction of an exponent ω' (working with all values of ω and ω') does not change the results (12).

It is instructive to consider the magnitude of the two correction terms appearing in equations (6). For example, the best fit with $\omega = \delta = 0.84$ and $\theta = 1.576$ gives $A_1 = -2.4499$ and $A_2 = 1.7977$ for the square lattice. The second term is of opposite sign to the first one, but there is no real cancellation between the two terms (Herrmann and Stauffer 1984, Privman 1984): for $N = 10$, say, the relative magnitude of the second term is small ($\sim 10\%$ of the first one). Still, it is important to take it into account and its omission changes the extrapolation significantly. For the triangular lattice the second correction is extremely small: for $\omega = \delta = 0.8$ one finds $A_1 = -1.5516$, $A_2 = 0.005\,30$ and the one-correction-term extrapolation is already satisfactory.

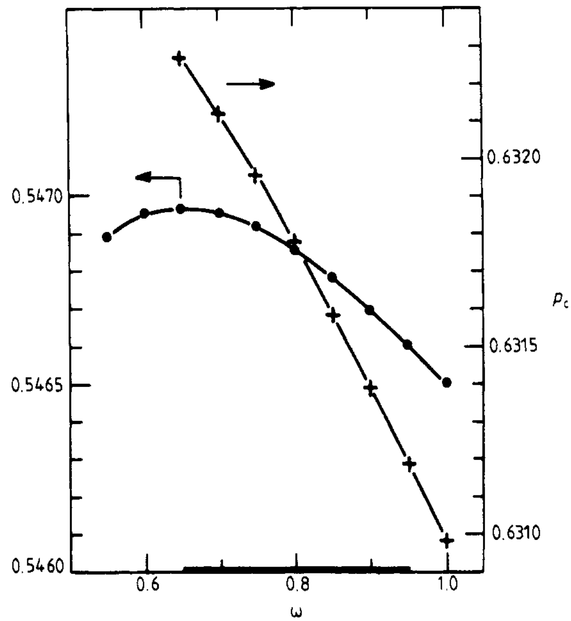

Figure 4. The extrapolated values of the threshold p_c obtained from equations (6*b*) and (8*b*) (using $N_i = 6$ and $N_f = 11$ in equation (11)) plotted against ω for the square (crosses) and triangular (circles) lattices. The scale on the right- (left-) hand side is for the square (triangular) lattice.

Let us finally compare our value for ω with the results obtained for the correction exponent Δ_1 in directed percolation from different approaches (Adler *et al* 1983). In our notation $\Delta_1 = \nu_\perp \omega$ and, using the quoted values $\Delta_1 = 0.95$–1.25 and $\nu_\perp = 1.1$, one obtains $\omega = 0.86$–1.14, consistent with our best estimate. This agreement should not be considered as conclusive in view of the difficulties encountered in related problems, for example for self-avoiding walks (Lyklema and Kremer 1985).

4. Conclusion

We have used various methods to analyse our results for partially directed percolation in order to extract accurate values of the percolation threshold from data obtained on finite strips by a transfer matrix technique. On the square lattice a simple extrapolation with one correction term gives unsatisfactory results for the anisotropy exponent, though the quality of the numerical fit is apparently very good. This shows that one must be cautious in using such extrapolations in the absence of extra information. Unfortunately, the leading correction exponent ω and the size $N_{\min}$ above which the simple asymptotic form of equation (10*a*) holds are generally unknown: Luck (1985) has studied these points recently and has concluded that predictions for them are likely to be more difficult than solving the model in the thermodynamic limit.

It is also very difficult to study large values of N, and the alternative approach we have found useful consists in working with several lattices and in using higher-order correction terms, as in equations (6). This approach is reasonable, even with rather small values of N (~ 10), because the condition that the extrapolated critical exponents and the correction exponents are the same for the different lattices reduces the number of parameters in the analysis; it leads to satisfactory results in our case, which was *a priori* a difficult one.

As a final remark, the leading correction exponent ω is clearly less than one for the partially directed percolation problems studied here. The rate of convergence is then rather slow, and this explains why higher correction terms are needed. The situation seems to be different in other problems (Derrida and De Seze 1982), but the present example is useful because it shows clearly the difficulties that may be encountered with the phenomenological renormalisation method.

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